
Assignment Questions

1. Let a be the last digit in your student number, b be the second-to-last digit in your student number, $c = 2a$, and $d = 3b$. For the functions f and g defined as: $f: \mathbb{Z} \rightarrow \mathbb{Z}; f(x) = ax + c$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}; g(x) = c^2 - d$, compute the following:

a. $f \circ g$

$$f(g(x)) = f(c^2 - d) = a(c^2 - d) + c$$

b. $g \circ f$

$$g(f(x)) = c^2 - d$$

c. $(f \circ g) \circ g$

$$f(g(g(x))) = f(g(c^2 - d)) = f(c^2 - d) = a(c^2 - d) + c$$

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2. Compute the closed form of the following series. As a clarifying example, the closed form for $\sum_{k=1}^n k$ is $\frac{n(n+1)}{2}$.

a. $\sum_{k=1}^n (1^k + 2)$

$$= \sum_{k=1}^n 3$$

$$= 3n$$

b. $\sum_{k=1}^n (4k - 3)$

$$= \sum_{k=1}^n 4k - \sum_{k=1}^n 3$$

$$= 4(\sum_{k=1}^n k) - 3n$$

$$= 2(k)(k+1) - 3n$$

$$= 2k^2 + 2k - 3n$$

c. $\sum_{k=1}^n (k^2 + 6k - 4)$

$$= \sum_{k=1}^n k^2 + \sum_{k=1}^n 6k - \sum_{k=1}^n 4$$

$$= \sum_{k=1}^n k^2 + 6 \sum_{k=1}^n k - \sum_{k=1}^n 4$$

$$= n(n+1)(2n+1)/6 + 6n(n+1)/2 - 4n$$

$$= n(n+1)(2n+1)/6 + 18n(n+1)/6 - 24n/6$$

$$= (2n^3 + 3n^2 + n + 18n^2 + 18n - 24n) / 6$$

$$= (2n^3 + 21n^2 - 5n) / 6$$

d. $\sum_{k=10}^{31} 5$

$$= 22(5) = 110$$

e. $\sum_{k=1}^n \sum_{j=1}^n 3(k - j)$

$$= 3(\sum_{k=1}^n (\sum_{j=1}^n k - j))$$

$$= 3(\sum_{k=1}^n (\sum_{j=1}^n k - \sum_{j=1}^n j))$$

$$= 3(\sum_{k=1}^n (nk - n(n+1)/2))$$

$$= 3((\sum_{k=1}^n nk) - (\sum_{k=1}^n n(n+1)/2))$$

$$= 3((\sum_{k=1}^n k) - (\sum_{k=1}^n n(n+1)/2))$$

$$= 3((n)(n)(n+1)/2 - (n)(n)(n+1)/2)$$

$$= 0$$

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$$\begin{aligned}
 \text{f. } & \sum_{k=1}^n \sum_{j=1}^n 6k \\
 &= 6(\sum_{k=1}^n \sum_{j=1}^n k) \\
 &= 6(\sum_{k=1}^n nk) \\
 &= 6n(\sum_{k=1}^n k) \\
 &= 6n(n)(n+1)/2 \\
 &= 3n(n)(n+1)
 \end{aligned}$$

3. Is the set of all positive integers divisible by 8 a countable set? Prove your answer.

Consider the function $f: \mathbb{Z}^+ \rightarrow \{e \mid e \text{ is divisible by } 8\}$, $f(n) = 8n$

Prove that $f(n)$ is injective:

$$f(x) = f(y)$$

$$8x = 8y$$

$$x = y$$

$\therefore f(n)$ is an injective function

Proof that $f(n)$ is surjective:

$$f(i) = j$$

(for arbitrary variable j)

$$f(i) = 8i$$

$$8i = j$$

$$i = j/8$$

$$\exists a \, f(a) = j$$

Existential Generalization

$$\forall b \, \exists a \, f(a) = b$$

Universal Generalization

$\therefore f(n)$ is a surjective function

$\therefore f(n)$ is a bijective function (since it is both injective and surjective)

\therefore the set S must be countable since a bijection from $\mathbb{Z}^+ \rightarrow S$ was found

Assignment Questions

4. Is the union of the set of all positive integers divisible by 5 with the set of all positive 3-digit integers a countable set? Prove your answer.

Consider the function

$$f: \mathbb{Z}^+ \rightarrow \{e \mid e \text{ is divisible by } 5\} \cup \{e \mid e \text{ is a three digit number}\}$$

$$\begin{aligned} f(n) = & 5n && \text{when } n < 20 \\ & n + 80 && \text{when } 20 \leq n \leq 899 \\ & 5n - 3500 && \text{when } n \geq 900 \end{aligned}$$

Prove that $f(n)$ is injective:

$$f(x) = f(y)$$

n.b.,

when $f(n)=5n$ the maximum possible value of the result of $f(n)$ is 95 because $f(n) = 5n$ only when $n < 20$

when $f(n)=n+80$ the min of $f(n)$ is 100 and the max of $f(n)$ is 979 because $f(n) = n+80$ only when $20 \leq n \leq 899$

when $f(n)=5n - 3500$ the min of $f(n)$ is 1000 because $f(n) = n+80$ only when $n \geq 900$

this means that the possible values of $f(n)$ can be expressed as:

$$f(x) < 20 \vee (100 \leq f(x) \leq 979) \vee f(x) > 1000$$

Proof by Cases:

Case 1:

$$f(x) < 20$$

$$f(x) = f(y)$$

$$f(y) < 20$$

$$5x = 5y$$

$$x = y$$

Case 2:

$$100 \leq f(x) \leq 979$$

$$f(x) = f(y)$$

$$100 \leq f(y) \leq 979$$

$$x+80 = y+80$$

$$x = y$$

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Case 3:

$$f(x) > 1000$$

$$f(x) = f(y)$$

$$f(y) > 1000$$

$$5x - 3500 = 5y - 3500$$

$$x = y$$

$\therefore f(n)$ is an injective function (because $f(x) = f(y) \rightarrow x = y$ in each case)

Proof that $f(n)$ is surjective:

let $j \in \{e \mid e \text{ is divisible by } 5\} \cup \{e \mid e \text{ is a three digit number}\}$

$$j \in \begin{aligned} &\{e \mid e \text{ is } < 100 \text{ and divisible by } 5\} \\ &\cup \{e \mid e \text{ is a three digit number}\} \\ &\cup \{e \mid e \text{ is } > 999 \text{ and divisible by } 5\} \end{aligned}$$

Proof by Cases:

Case 1:

$$f(i) = j$$

$$f(i) = 5i$$

$$5i = j$$

$$i = j/5$$

$(j \in \{e \mid e \text{ is } < 100 \text{ and divisible by } 5\})$
(for arbitrary variable j)

Case 2:

$$f(i) = j$$

$$f(i) = i+80$$

$$i+80 = j$$

$$i = j-80$$

$(j \in \{e \mid e \text{ is a three digit number}\})$
(for arbitrary variable j)

Case 3:

$$f(i) = j$$

$$f(i) = 5i - 3500$$

$$5i - 3500 = j$$

$$i = (j+3500) / 5$$

$(j \in \{e \mid e \text{ is } > 999 \text{ and divisible by } 5\})$
(for arbitrary variable j)

$$\exists a \, f(a) = j$$

$$\forall b \, \exists a \, f(a) = b$$

$\therefore f(n)$ is a surjective function

Existential Generalization

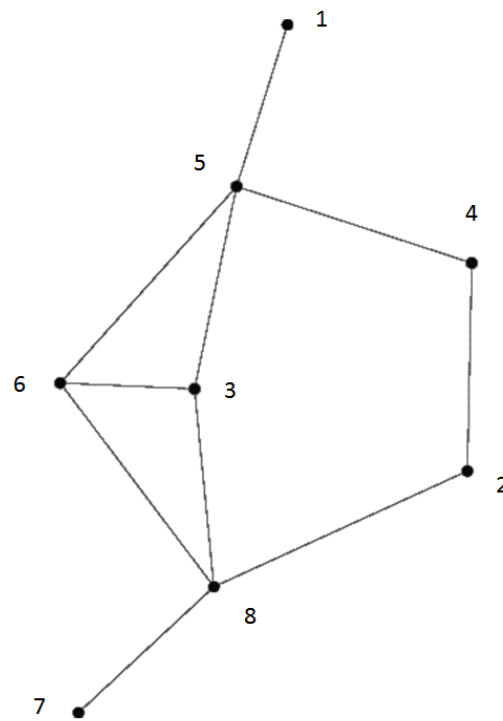
Universal Generalization

Assignment Questions

$\therefore f(n)$ is a bijective function (since it is both injective and surjective)

\therefore the set S must be countable since a bijection from $\mathbb{Z}^+ \rightarrow S$ was found

Questions 5-10 all refer to the undirected graph specified below:



5. If the graph above is denoted G where $G = (V, E)$, what are the values of V and E ?

$V = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$E = \{ \{1, 5\}, \{2, 4\}, \{2, 8\}, \{3, 5\}, \{3, 6\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{6, 8\}, \{7, 8\} \}$

Assignment Questions

6. Provide an adjacency matrix representation of this graph.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| 4 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 5 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 6 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 8 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 |

7. Provide an adjacency list representation of this graph.

1: 5
 2: 4, 8
 3: 5, 6, 8
 4: 2, 5
 5: 1, 3, 4, 6
 6: 3, 5, 8
 7: 8
 8: 2, 3, 6, 7

8. Transform this undirected graph into a directed graph G' by changing each undirected edge into a directed edge that moves away from vertex 1. Provide the values of V' and E' such that $G' = (V', E')$ and please note that there is more than one correct solution to this question.

$V' = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$E' = \{ (1, 5), (5, 3), (5, 4), (5, 6), (3, 6), (3, 8), (4, 2), (6, 8), (2, 8), (8, 7) \}$

Assignment Questions

9. Provide a subgraph G'' of the graph above that is an element of the set K_n . Specify this graph by providing listing the elements of both V'' and E'' such that $G'' = (V'', E'')$ and then provide an adjacency matrix representation of this graph as well.

Any of the following would constitute a correct answer!

| | |
|---------------------------|--|
| $K_3 @ V'' = \{3, 5, 6\}$ | $E'' = \{\{3, 5\}, \{5, 6\}, \{3, 6\}\}$ |
| $K_3 @ V'' = \{3, 6, 8\}$ | $E'' = \{\{3, 6\}, \{6, 8\}, \{3, 8\}\}$ |
| $K_2 @ V'' = \{1, 5\}$ | $E'' = \{\{3, 6\}, \{6, 8\}, \{3, 8\}\}$ |
| $K_2 @ V'' = \{1, 5\}$ | $E'' = \{\{1, 5\}\}$ |
| $K_2 @ V'' = \{2, 4\}$ | $E'' = \{\{2, 4\}\}$ |
| $K_2 @ V'' = \{2, 8\}$ | $E'' = \{\{2, 8\}\}$ |
| $K_2 @ V'' = \{3, 5\}$ | $E'' = \{\{3, 5\}\}$ |
| $K_2 @ V'' = \{3, 6\}$ | $E'' = \{\{3, 6\}\}$ |
| $K_2 @ V'' = \{3, 8\}$ | $E'' = \{\{3, 8\}\}$ |
| $K_2 @ V'' = \{4, 5\}$ | $E'' = \{\{4, 5\}\}$ |
| $K_2 @ V'' = \{5, 6\}$ | $E'' = \{\{5, 6\}\}$ |
| $K_2 @ V'' = \{6, 8\}$ | $E'' = \{\{6, 8\}\}$ |
| $K_2 @ V'' = \{7, 8\}$ | $E'' = \{\{7, 8\}\}$ |
| $K_1 @ V'' = \{1\}$ | $E'' = \emptyset$ |
| $K_1 @ V'' = \{2\}$ | $E'' = \emptyset$ |
| $K_1 @ V'' = \{3\}$ | $E'' = \emptyset$ |
| $K_1 @ V'' = \{4\}$ | $E'' = \emptyset$ |
| $K_1 @ V'' = \{5\}$ | $E'' = \emptyset$ |
| $K_1 @ V'' = \{6\}$ | $E'' = \emptyset$ |
| $K_1 @ V'' = \{7\}$ | $E'' = \emptyset$ |
| $K_1 @ V'' = \{8\}$ | $E'' = \emptyset$ |

10. Consider the smallest set of edges E''' that could need to be added to the graph above such that the graph $(V, E \cup E''')$ contains W_6 . Provide an exhaustive list of the elements of E''' and specify the cardinality of E''' .

$$E''' = \{\{3, 4\}, \{2, 3\}\}$$

$$|E'''| = 2$$

Assignment Questions

Questions 11-12 on this page all refer to the undirected graph specified below:

11. Given the graph $G = (V, E)$ in the figure above, compute the breadth-first search tree starting from the vertex that is labelled with the third-to-last digit in your student number. To clarify, if your student number is 100123456, start your breadth-first search tree from vertex 4. You must provide your search tree as an adjacency list; do not "draw" your search trees. Whenever you have a "choice" of which adjacent vertex to consider, you must consider them in numerical order from least to greatest.

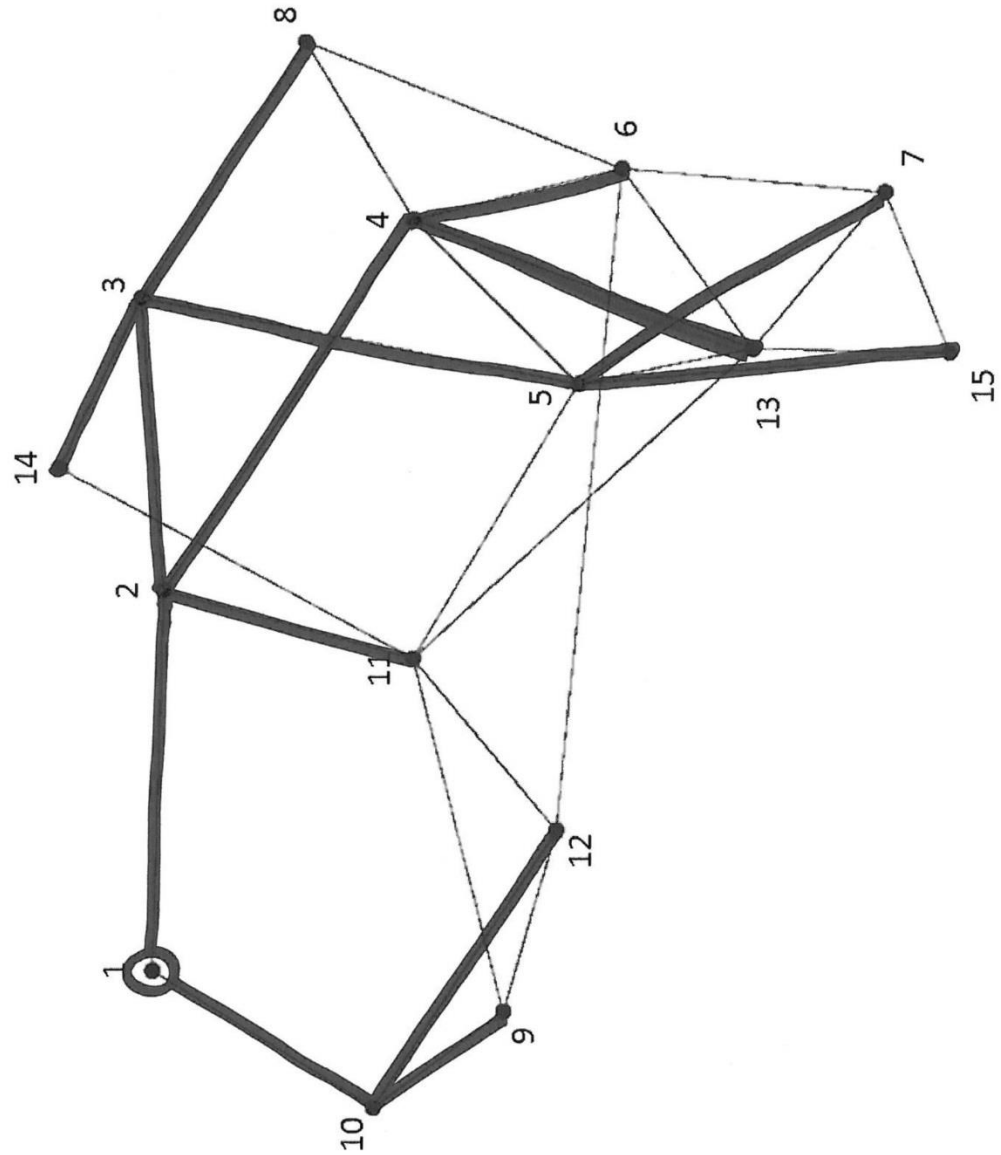
Every possible solution to this problem is included below.

12. Given the graph $G = (V, E)$ in the figure above, compute the depth-first search tree starting from the vertex that is labelled with the fourth-to-last digit in your student number. To clarify, if your student number is 100123456, start your depth-first search tree from vertex 3. You must provide your search tree as an adjacency matrix; do not "draw" your search trees. Whenever you have a "choice" of which adjacent vertex to consider, you must consider them in numerical order from least to greatest.

Every possible solution to this problem is included below.

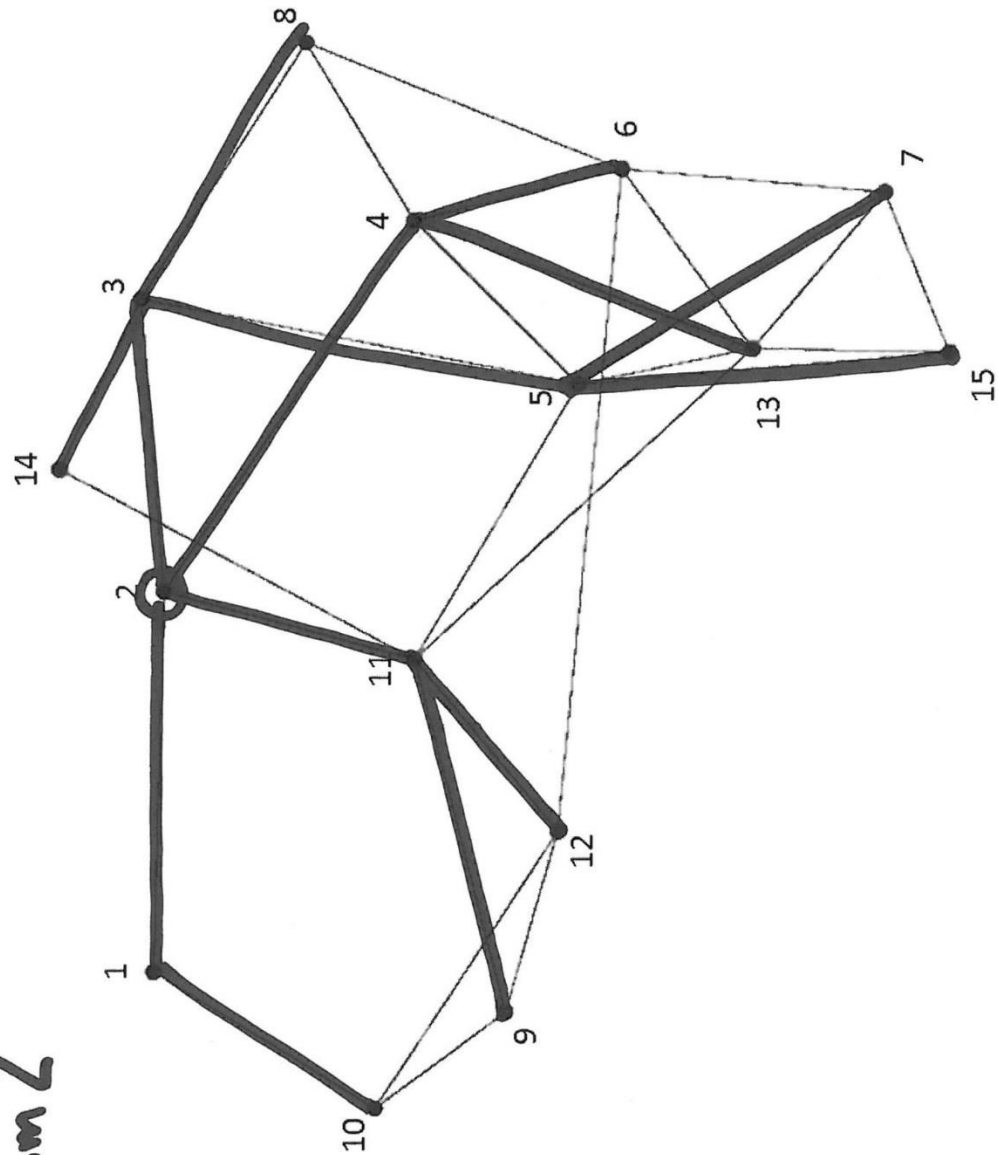
Assignment Questions

BFS from 1



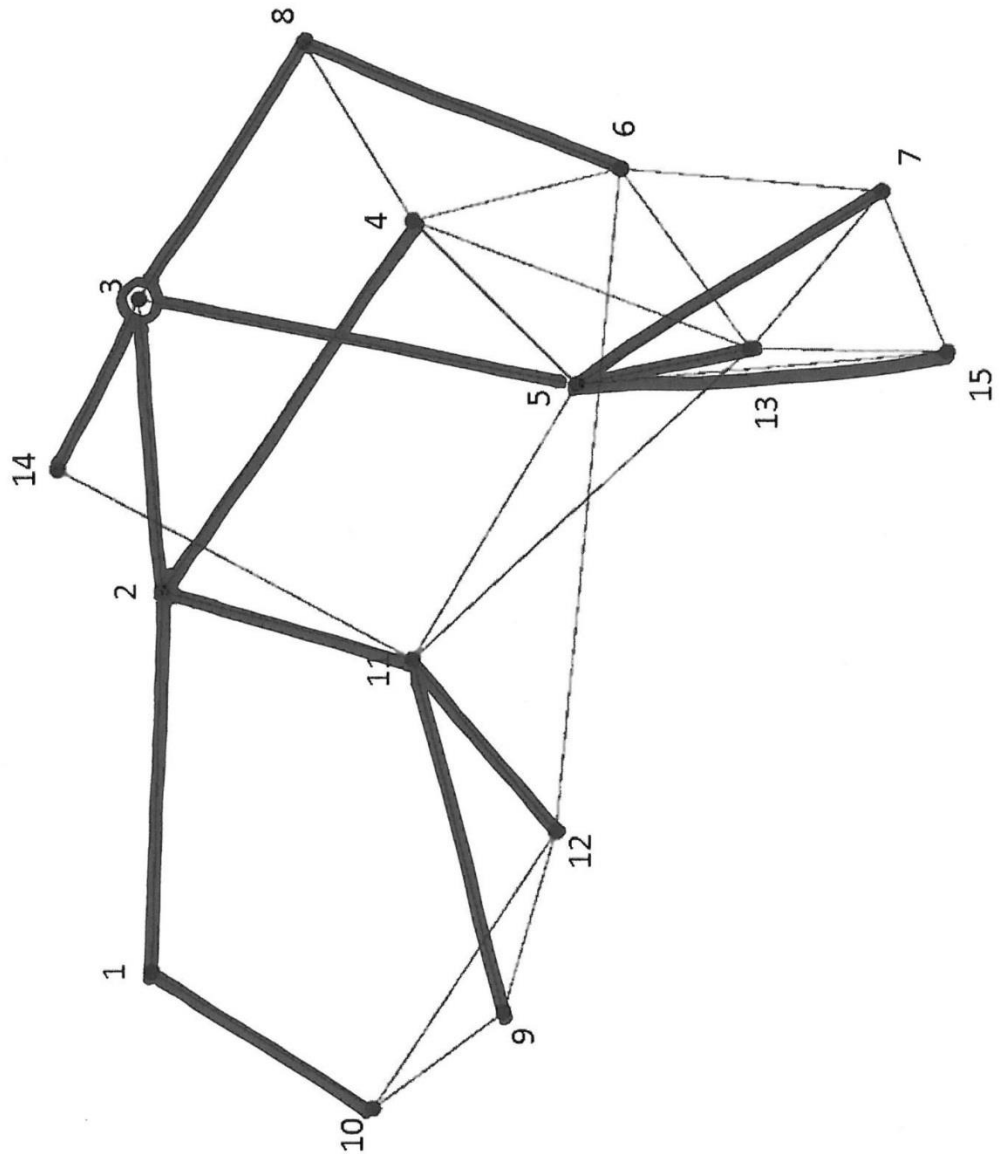
Assignment Questions

~~Handwritten~~ BFS from 2



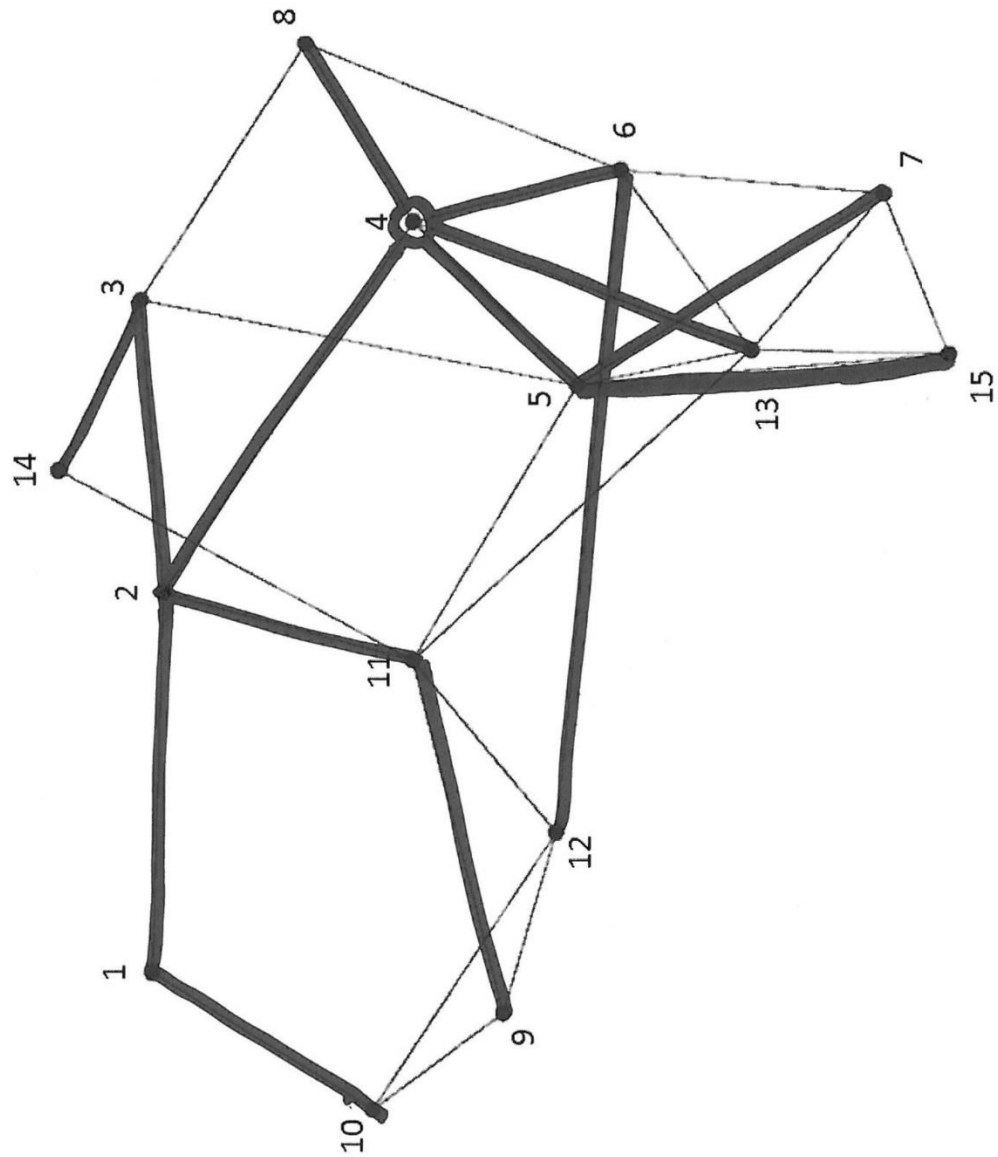
Assignment Questions

BFS from 3



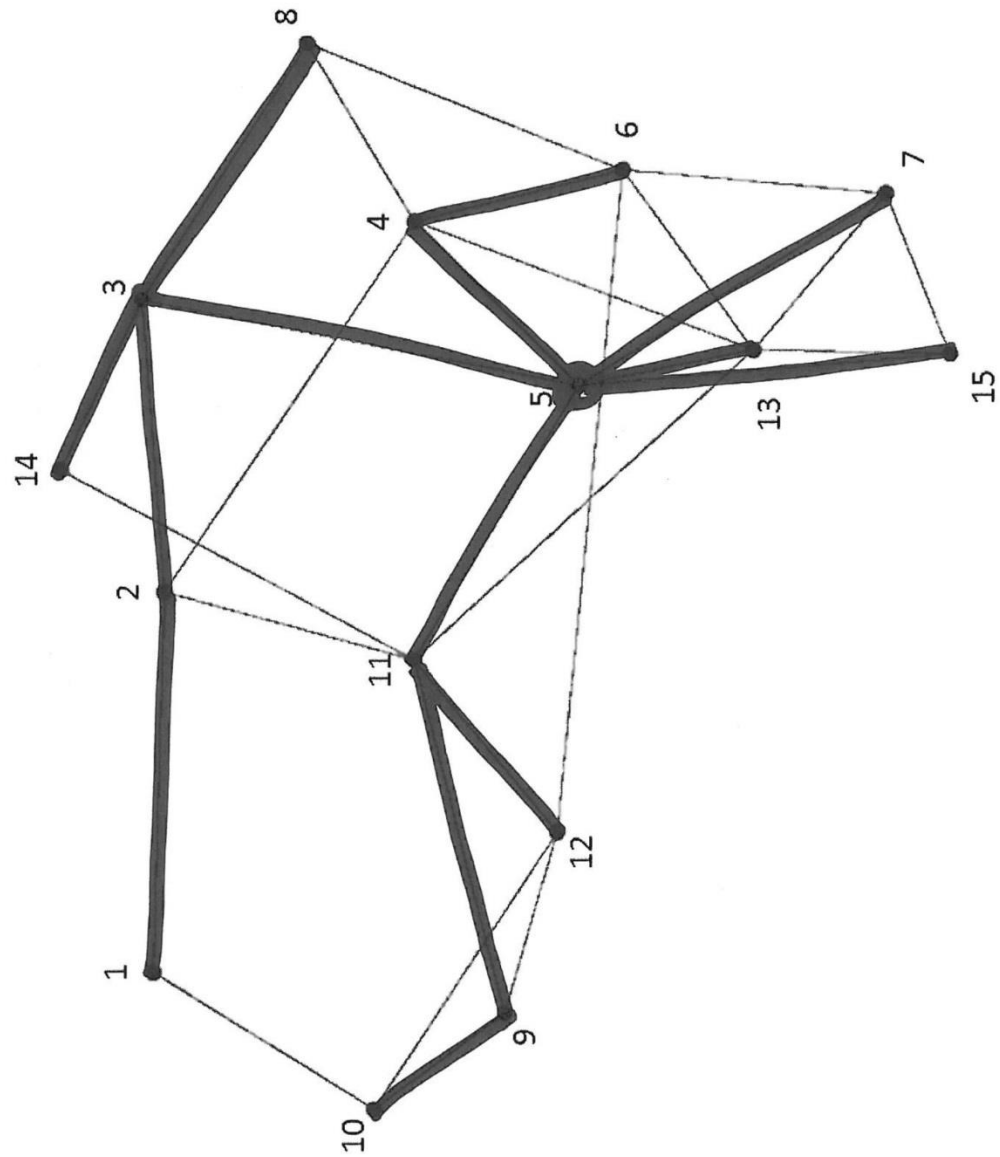
Assignment Questions

BFS from 4



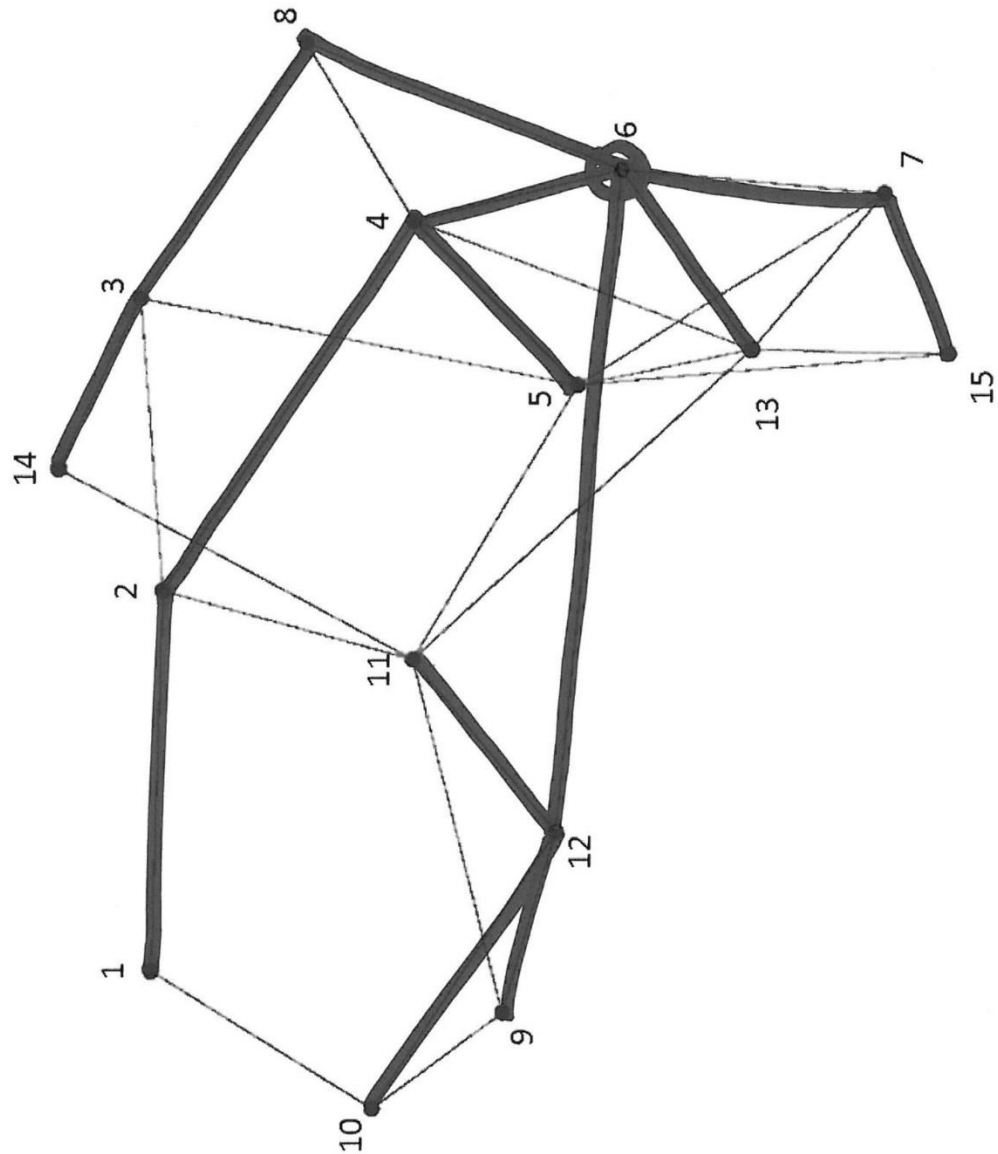
Assignment Questions

BFS from 5



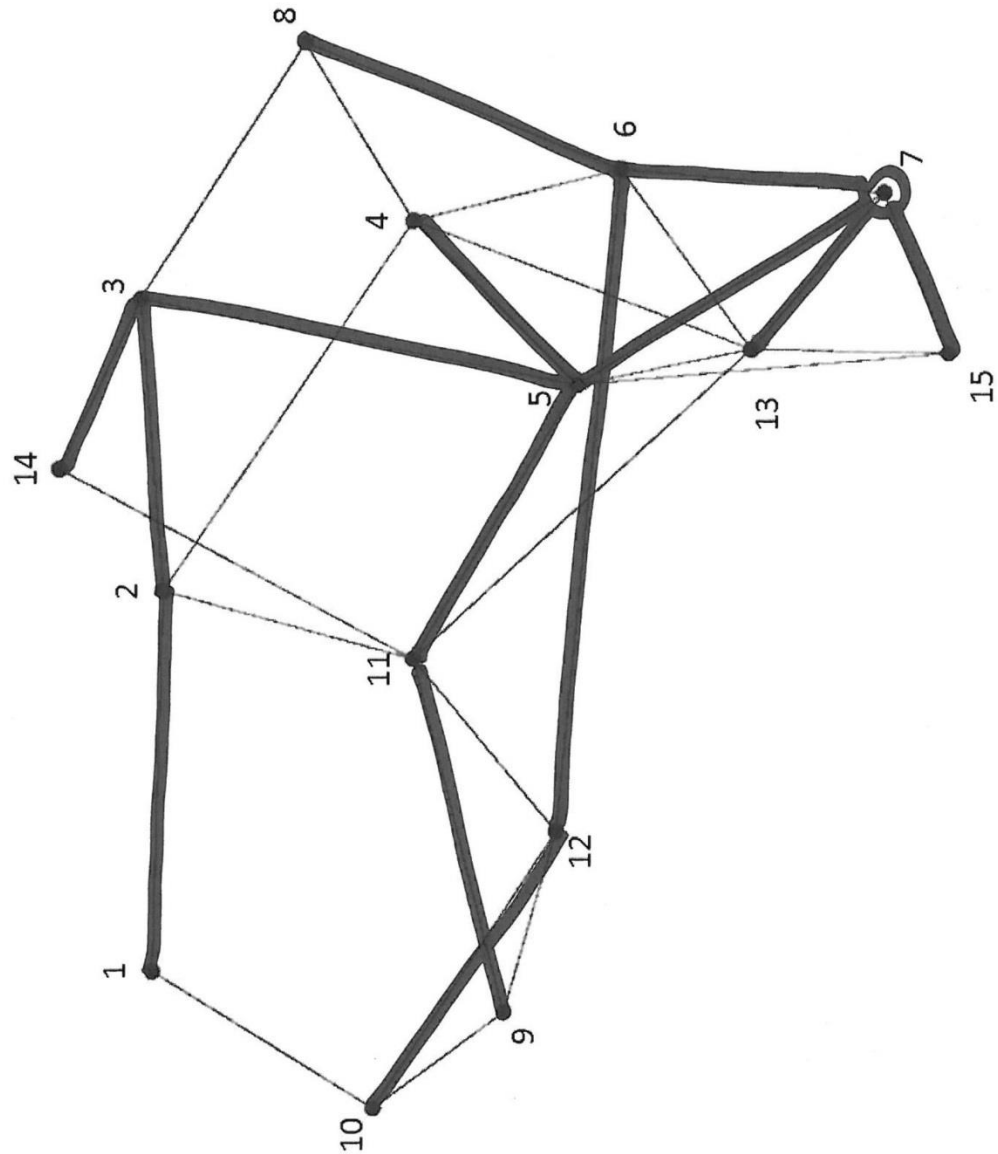
Assignment Questions

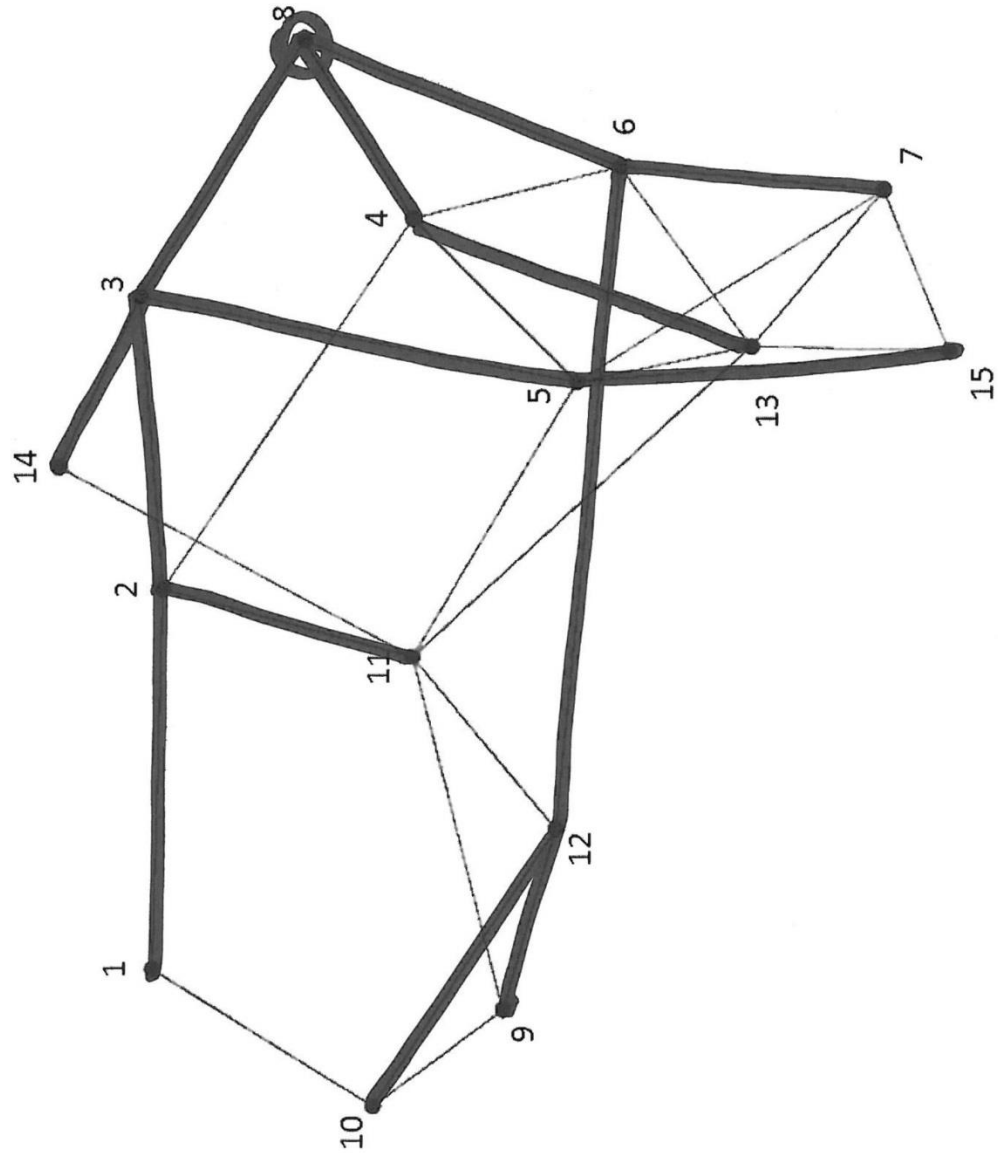
BFS from 6



Assignment Questions

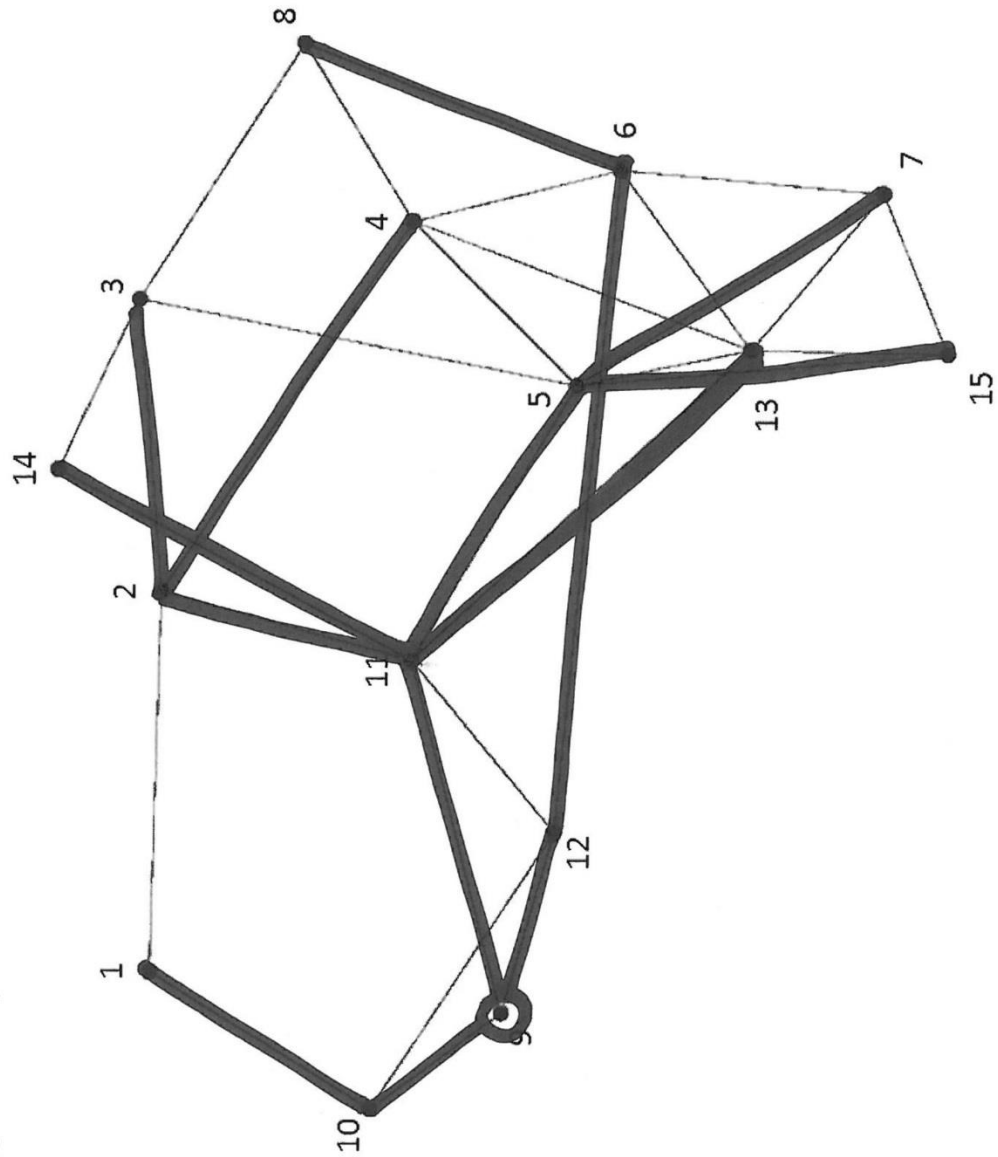
BFS from 7





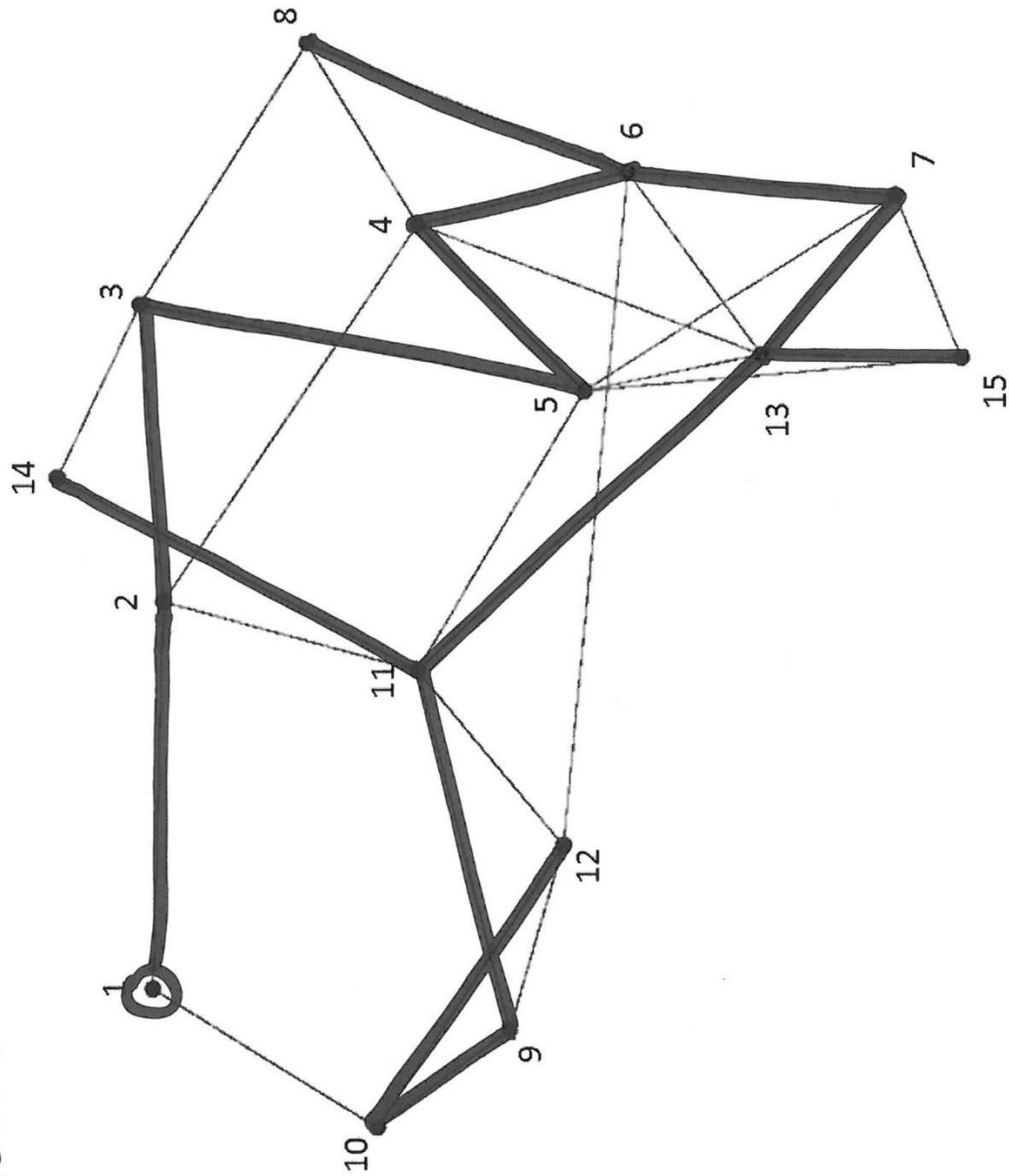
Assignment Questions

BFS from 9



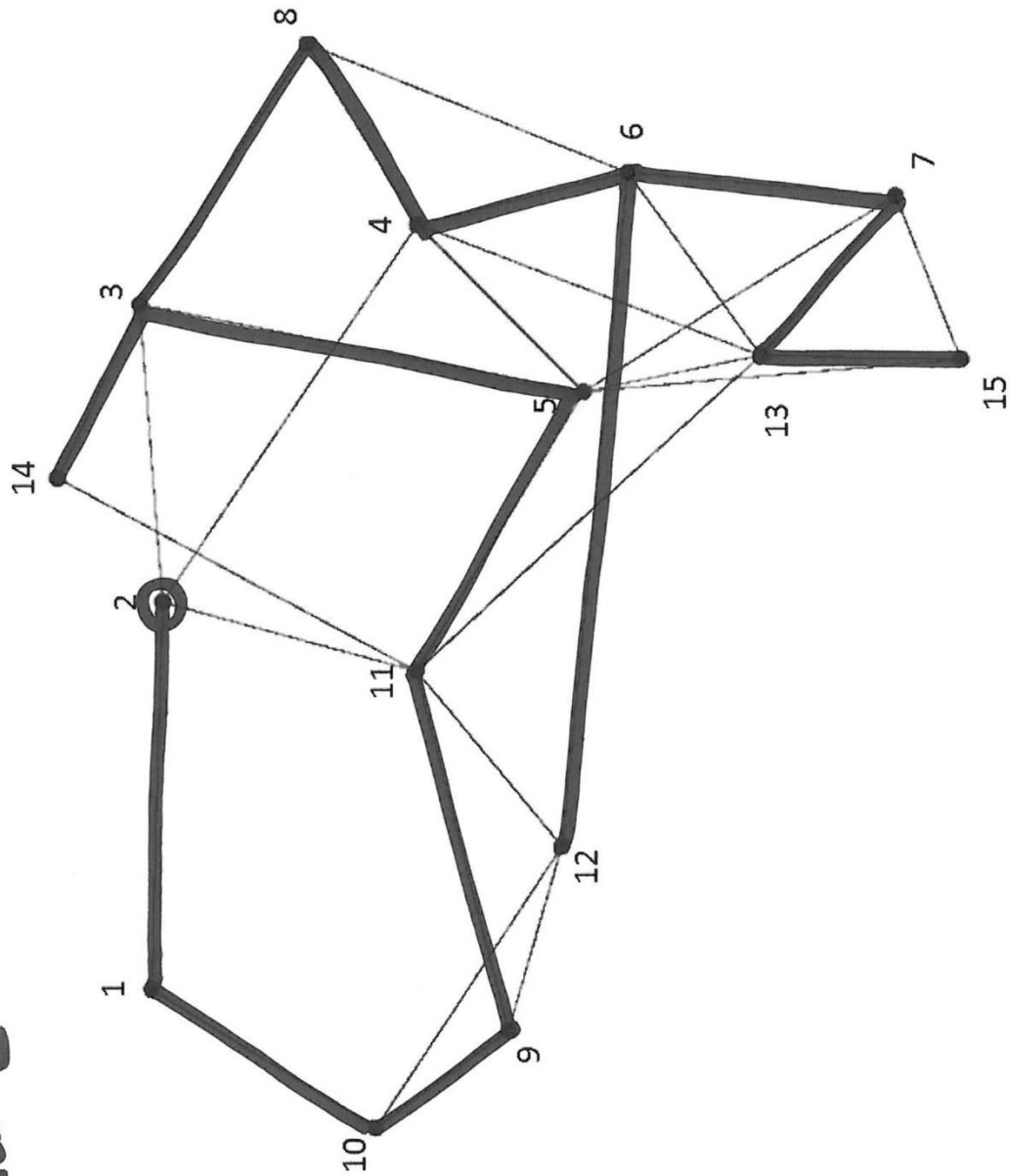
Assignment Questions

DFS from 1



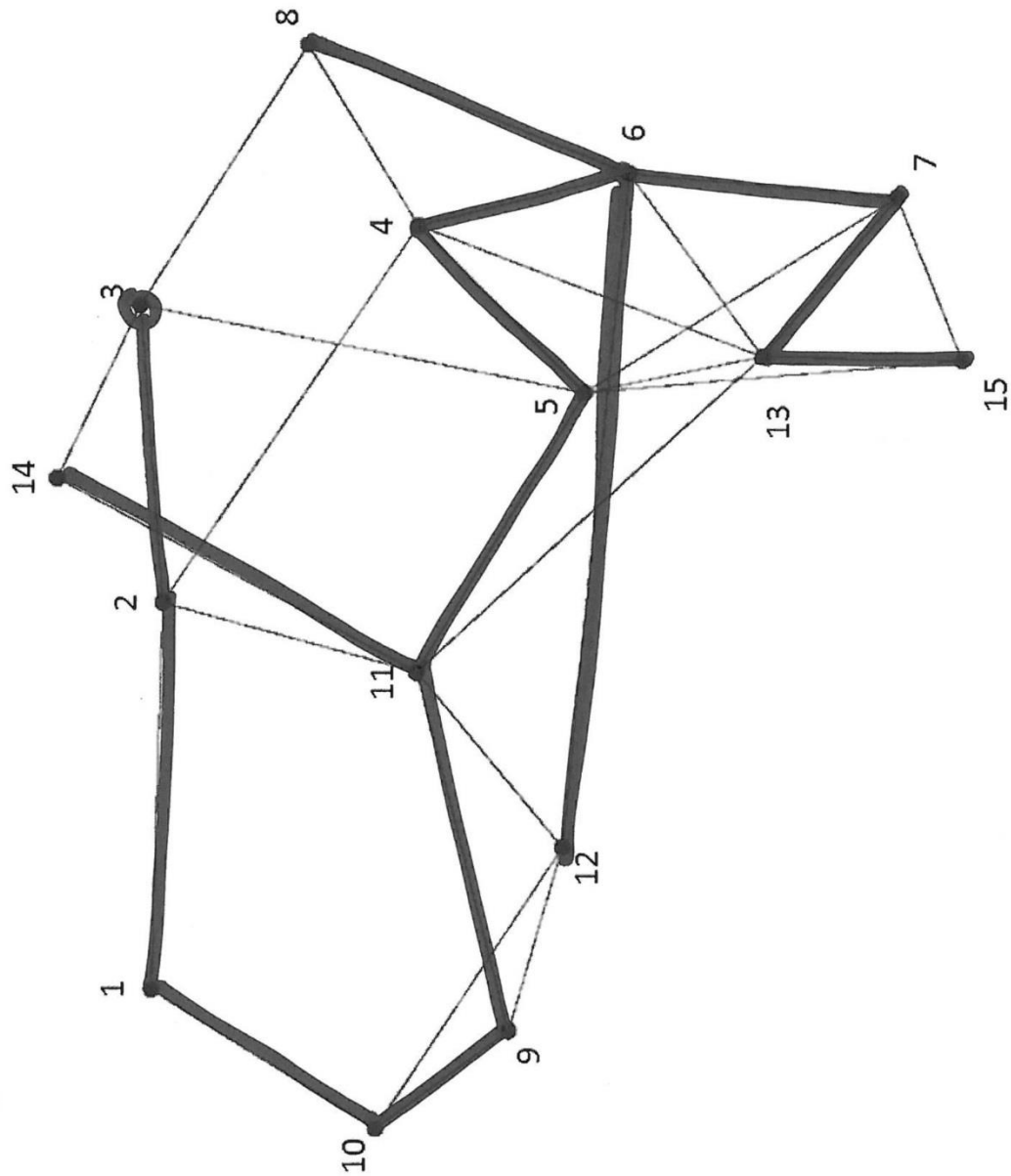
Assignment Questions

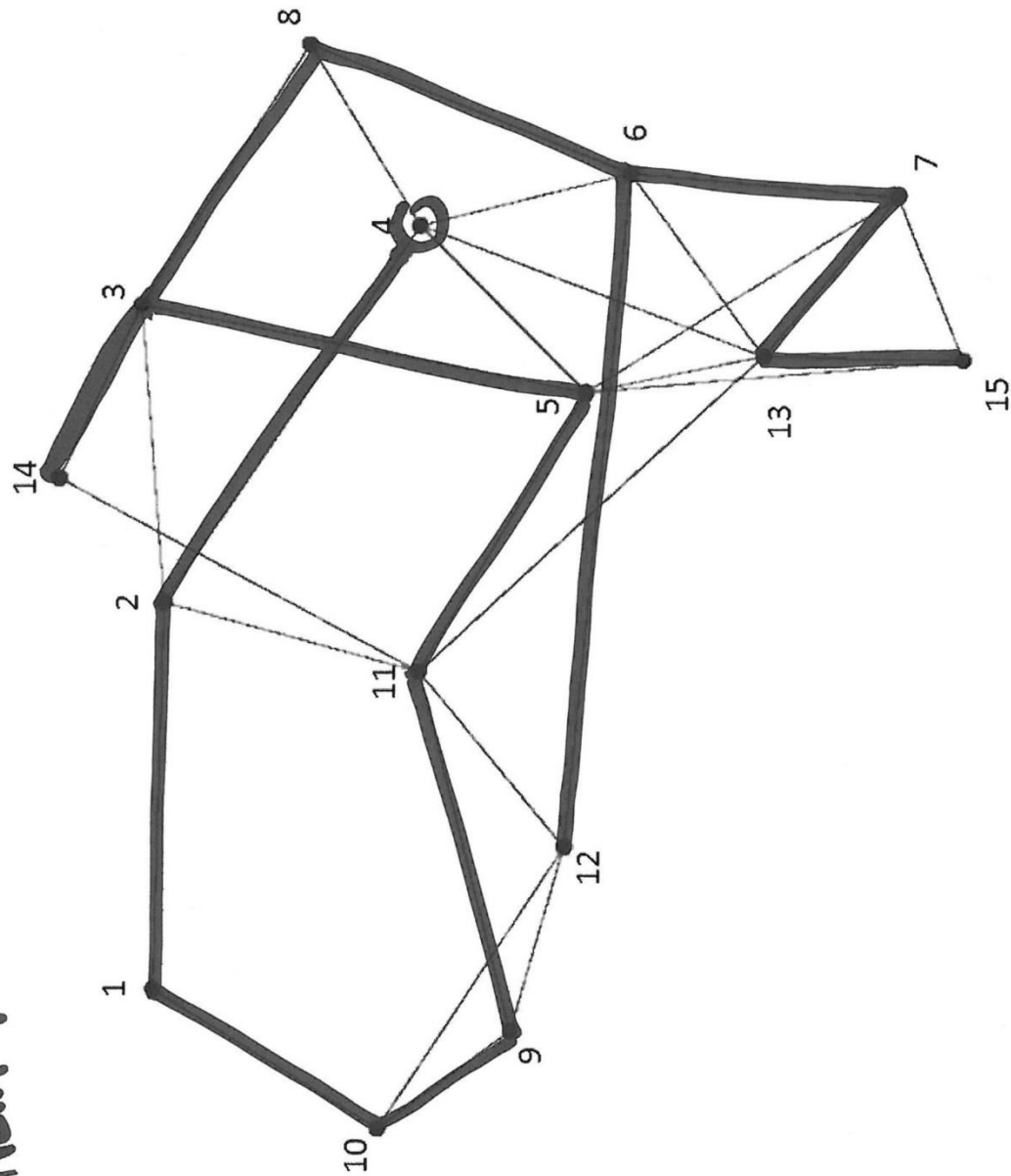
DFS from 2



Assignment Questions

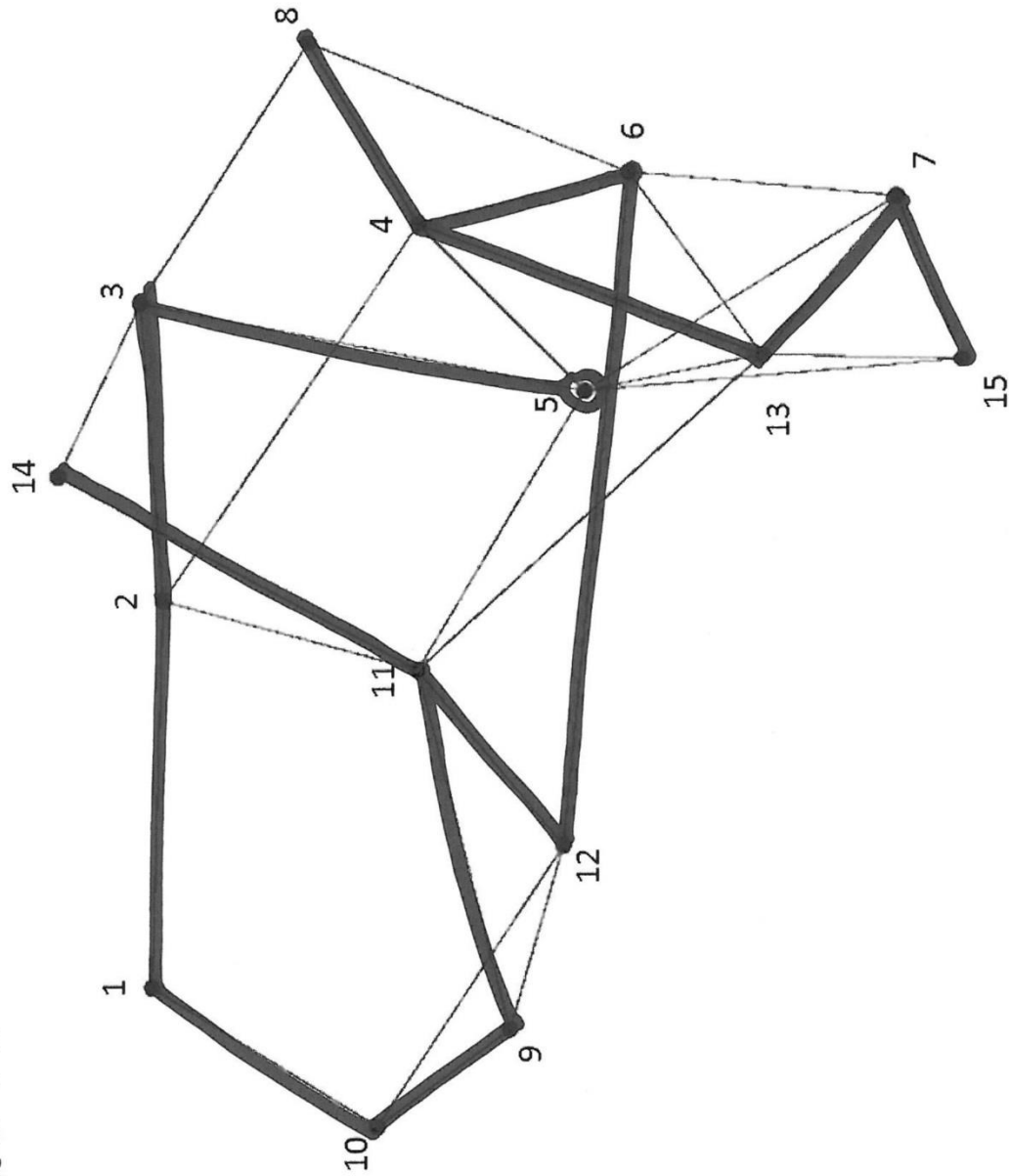
DFS from 3

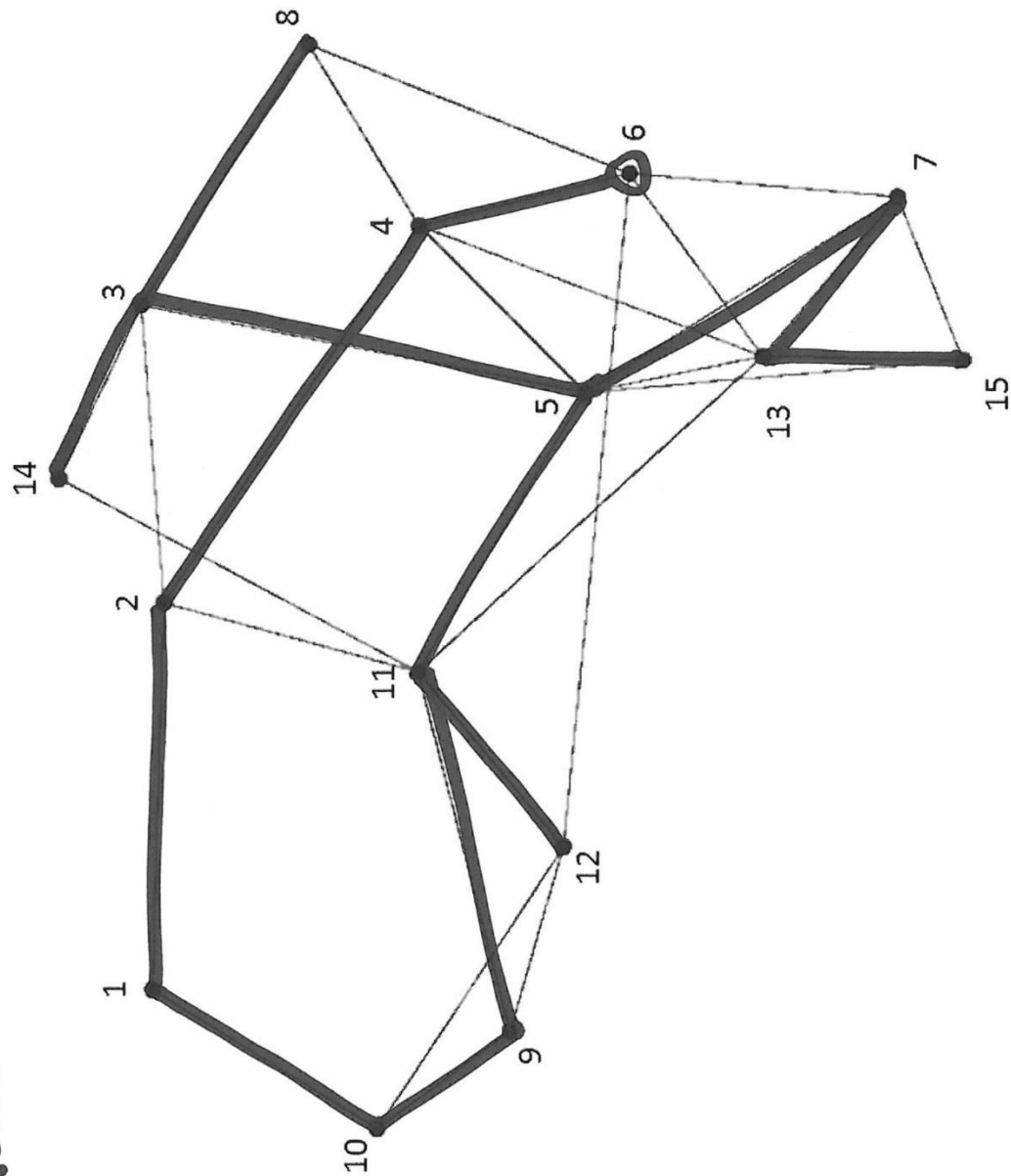




Assignment Questions

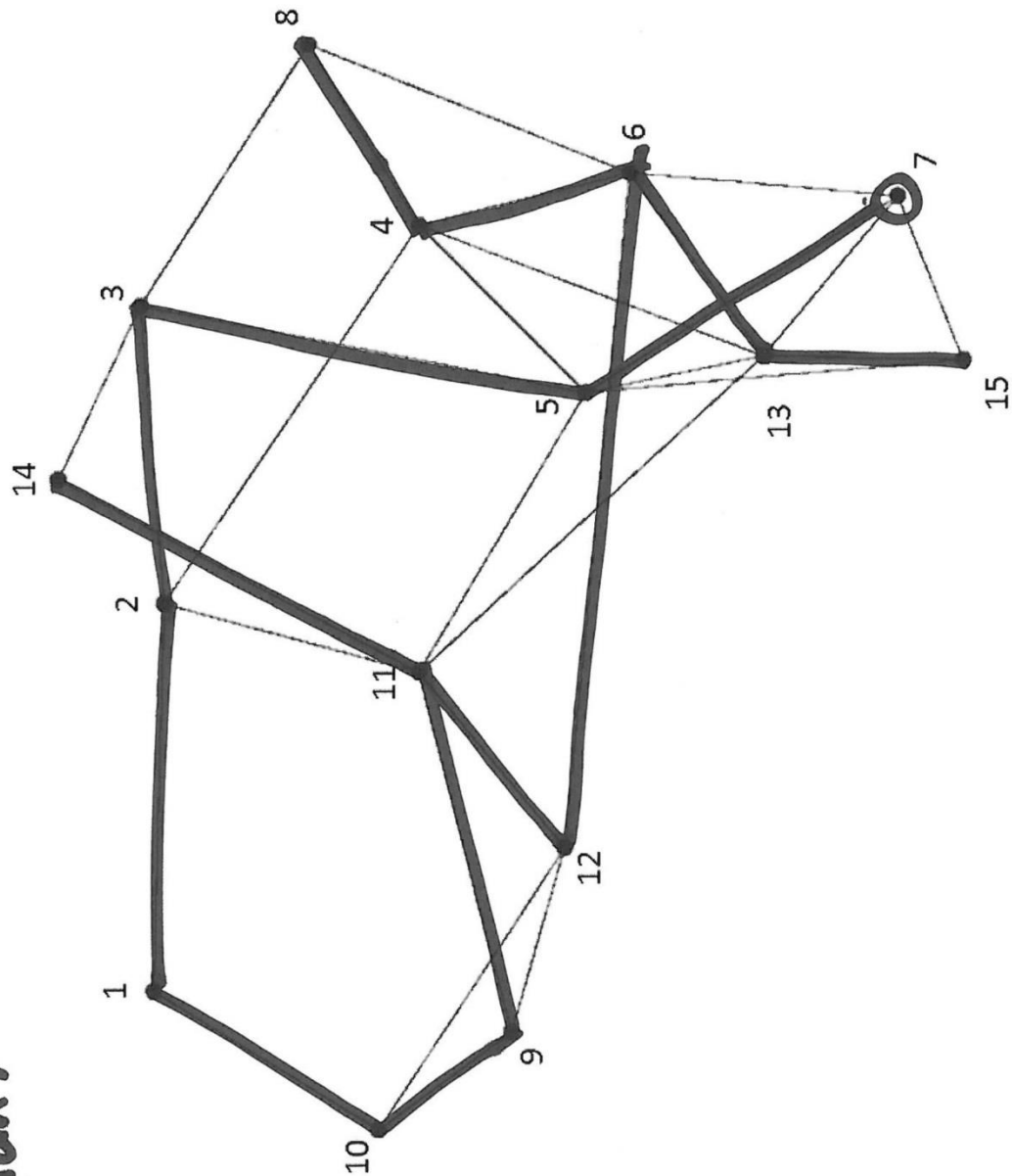
DFS from 5





Assignment Questions

DFS from 7



Assignment Questions

DFS from 8

